Ch 5 Continuous Functions

Def $f : D \to \mathbb{R}$ is said to be continuous (cts) at $x_0 \in D$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

(1) $|f(x) - f(x_0)| < \varepsilon, \ \forall x \in V_\delta(x_0) \cap D$

[equivalently, for the last line

(1*) $|f(x) - f(x_0)| < \varepsilon, \ \forall x \in V_\delta(x_0) \cap D \setminus \{x_0\}$)]

Remarks

1. If $x_0 \in D$ is, additionally, an cluster-point w.r.t. D then f is cts at x_0 iff
$$\lim_{x \to x_0} f(x) = f(x_0).$$

2. If $x_0 \in D$ is, additionally, an isolated point w.r.t. D (so $\exists \ \delta_0 > 0$ s.t. $V_\delta(x_0) \cap D = \{x_0\}$) then (1) always holds provided that $0 < \delta \le \delta_0$
so any function $f : D \to \mathbb{R}$ is always cts at an <u>isolated</u> point x_0 of D. (provided that $x_0 \in D$).

Th (5.1.3) Let $f: D \to \mathbb{R}$, $x_0 \in D$. Then the FSAE:

(i) f is cts at x_0

(ii) $\lim f(x_n) = f(x_0)$ whenever (x_n) is a seq in D with $x_n \to x_0$.

Proof. Note that, if $x_0 \in D$ is isolated pt w.r.t. D, then both (i) and (ii) are true because ~~then~~, whenever (x_n) is a seq in D with $x_n \to x_0$, then $\exists\ N \in \mathbb{N}$ s.t. $x_n = x_0\ \forall\ n \geq N$.

Thus, one can assume henceforth that $x_0 \in D$ is cluster-point w.r.t. D. Then the proof can be achieved similarly from the proof given in the preceding chapter [Ex!]

§ 5.2 Combination of Cts Functions

The continuity property at $x_0 \in D$ of functions is "stable" with respect to operations:

$+$, $-$, $\times$

$\div$ (provided that the function in the donominator does not take zero-value at any point of D)

absolute-value function & lattice-operations

Th (5.2.5). Let $f : D \to [0, \infty)$ be cts at $x_0 \in D$. Then $\sqrt{f}$ (defined by $(\sqrt{f})(x) = \sqrt{f(x)}\ \forall x \in D$) is also cts at x_0.

Pf. Separately consider 2-cases:

(i) $f(x_0) = 0$. Let $\varepsilon > 0$. Then $\exists\, \delta > 0$ s.t.

$$f(x) = |f(x) - f(x_0)| < \varepsilon^2 \quad \forall x \in D \cap V_\delta(x_0)$$

and so

$$\sqrt{f(x)} = \left|\sqrt{f(x)} - \sqrt{f(x_0)}\right| < \varepsilon \quad \forall x \in D \cap V_\delta(x_0).$$

(ii) Case that $f(x_0) \neq 0$. Let $\varepsilon > 0$. Then $\exists\, \delta > 0$ such that

$$|f(x) - f(x_0)| < \sqrt{f(x_0)} \cdot \varepsilon \quad \forall x \in D \cap V_\delta(x_0).$$

Note then that, $\forall x \in D \cap V_\delta(x_0)$ one has

$$\left|\sqrt{f(x)} - \sqrt{f(x_0)}\right| = \frac{|f(x) - f(x_0)|}{\sqrt{f(x)} + \sqrt{f(x_0)}} \leq \frac{|f(x) - f(x_0)|}{\sqrt{f(x_0)}} < \frac{\sqrt{f(x_0)}\, \varepsilon}{\sqrt{f(x_0)}} = \varepsilon$$

Th (5.2.6)

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & \mathbb{R} \\ \cup & & \cup & & \\ a_0 & & b_0 = f(a_0) & & \end{array}$$

Suppose f is cts at a_0 & g is cts at $b_0 = f(a_0)$. Then $g \circ f$ is cts at a_0.

§5.3 Continuous Functions of Intervals

The most important tool for our investigation (in the present & the next sections) is the B-W th:

① each bounded seq has a convergent subsequence

② for each bounded seq (x_n) in $[a,b]$ (with reals $a<b$), $\exists$ a subseq (x_{n_K}) convergent to some $x_0 \in [a,b]$.

(B-W theo + order preserving)

Th (5.3.2) + 5.3.4 (global "Boundedness Th & Max-Min Th) Let $I = [a,b]$ be a bounded closed interval and $f: I \to \mathbb{R}$ be continuous (in the sense that f is cts at each point of I). Then

(i) f is bounded on I ; $\exists 0<M \in \mathbb{R}$ s.t. $|f(x)| \leq M \; \forall x \in I$

(ii) $\exists \, x_*, x^{**} \in I$ s.t. $\quad f(x_*) \leq f(x) \leq f(x^{**}) \; \forall x \in I$.

(i.e. f "attains" max. value and min value)

Remark. (ii) is actually a stronger result than (i).

Proof (i). Suppose not. Then $\forall n \; \exists \; x_n \in I$ s.t. $|f(x_n)| > n$. By ② above, $\exists$ a subseq and some $x_0 \in [a,b)$ s.t. $x_{n_K} \to x_0$. Since x_0 is a continuity pt of f, it follows that $f(x_{n_K}) \to f(x_0) \in \mathbb{R}$ (?) Consequently the seq $(f(x_{n_K}))_{K \in \mathbb{N}}$ is bounded (??) But this is not the case as, by our choice of x_{n_K}, one has

$$|f(x_{n_K})| > n_K \geq K \quad \forall K \in \mathbb{N}.$$

(ii). Let
$$f(I) := \{ f(x) : x \in I \}.$$

Then (i), $f(I)$ is bounded (above and below) and hence

$s_* \triangleq \inf f(I)$ & $s^* \triangleq \sup f(I)$ exist in $\mathbb{R}$ (by what ???). By def of "sup", $\forall n \in \mathbb{N}$, $\exists\, x_n \in I$ s.t. $s^* - \frac{1}{n} < f(x_n) \leq s^*$. By ② again, $\exists$ a subseq (x_{n_k}) and $x_0 \in I$ s.t. $x_{n_k} \to x_0$; and, as before one then knows that $f(x_{n_k}) \to f(x_0)$. Passing to the limits in

$$s^* - \frac{1}{n_k} < f(x_{n_k}) \leq s^*$$

the order-preserving tells us that $s^* \leq f(x_0) \leq s^*$ so $f(x) \leq s^* \leq f(x_0)$ $\forall\, x \in I$ (so x_0 has the required property for x^*).

Similarly one can do for minimum pts.

Th (5.3.5 + 5.3.7). Let $f: I \to \mathbb{R}$ be cts with $I = [a,b]$ (bd. & closed).

(i) If $f(a)f(b) < 0$ then $\exists\, c \in (a,b)$ s.t. $f(c) = 0$

(ii) If k lies between $f(a)$ & $f(b)$ $\left[\begin{array}{l} f(a) < k < f(b) \text{ or} \\ f(a) > k > f(b) \end{array}\right]$ then $\exists\, c \in (a,b)$ s.t. $f(c) = k$.

Proof Need only show (i) [for (ii), one considers $g(x) := f(x) - k$ $\forall\, x \in I$ & applies (i) to g in place of f].

For (i), we assume w.l.g that $f(a) < 0 < f(b)$ [Why?] Write $I_1 = [a_1, b_1] = [a,b]$. Unless $f\left(\frac{a_1+b_1}{2}\right) = 0$ (proving (i)), $\exists\, I_2 = [a_2, b_2]$ with $l(I_2) := b_2 - a_2 = \frac{l(I_1)}{2} \left(= \frac{b-a}{2}\right)$ s.t

$f(a_2) < 0 < f(b_2)$ $\{$ if $f(\frac{a_1+b_1}{2}) < 0$ then define
$I_2 := [\frac{a_1+b_1}{2}, b_1]$; if $f(\frac{a_1+b_1}{2}) > 0$ then define
$I_2 = [a_1, \frac{a_1+a_2}{2}]$ $\}$.

Inductively, unless one already gets $c \in (a,b)$ s.t. $f(c)=0$, one has a nested seq (I_n) with each $I_n = [a_n, b_n]$, $l(I_n) = \frac{b-a}{2^{n-1}}$ such that $f(a_n) < 0 < f(b_n)$. By the nested interval theo

$\exists\ c \in \bigcap_{n \in \mathbb{N}} I_n$ (so $a_n \leq c \leq b_n\ \forall n$). Then $c \in [a,b]$

$0 \leq b_n - c \leq b_n - a_n = \frac{(b-a)}{2^{n-1}} \to 0$ and so $\lim_n b_n = c$

By continuity of f at c it follows that $\lim_n f(b_n) = f(c)$ so $f(c) \geq 0$ (by the order-preserving as $f(b_n) > 0\ \forall n$)

Similarly $f(c) = \lim_n f(a_n) \leq 0$ and so $f(c) = 0$.

Remark. Try to use the Monotone Conv. Th.

Cor1. Let I be any interval and let $f: I \to \mathbb{R}$ be cts. Then $f(I)$ is an interval.

pf. By the preceding theorem, $f(I)$ is "order-convex"

$y_1, y_2 \in f(I)$, $y_1 < z < y_2 \implies z \in f(I)$.

By interval characterization theorem $f(I)$ is an interval.

Cor 2. Let $I = [a,b]$ be a bounded closed interval and $f: I \to \mathbb{R}$ cts. Then $f(I)$ is also a bounded closed interval. In fact $f(I) = [m, M]$, where

$M := f(x^*)$ and $m := f(x_*)$ with min. pt x_* and max. pt of f on $[a,b]$.

Possible extension. Let $f: [a,\infty) \to \mathbb{R}$ be cts and suppose that $f(x_0) > l = \lim_{x\to+\infty} f(x) \in \mathbb{R}$ for some $x_0 \in [a,\infty)$. Then f attains its global max at some point $x^* \in [a,\infty)$.

pf. By assumption, $\exists\ b > x_0$ such that

$$|f(x) - l| < f(x_0) - l \quad \forall\ x \geqslant x_0 \qquad (*)$$

(so

$$f(x) < f(x_0) \qquad \forall\ x \geqslant x_0\ )$$

By max-min value theo, $\exists\ x^* \in [a,b]$ such that

$$f(x) \leqslant f(x^*) \quad \forall\ x \in [a,b] \qquad (1)$$

Since $x_0 \in [a,b]$, one has also that

$$f(x) < f(x_0) \leqslant f(x^*) \quad \forall\ x > b$$

thanks to $(*)$. Combining this with (1), we see that $f(x) \leqslant f(x^*)$ $\forall\ x \in \underbrace{[a,b] \cup (b,+\infty)}_{= [a,\infty)}$.